MATHEMATICAL MODELS OF STRESS RELAXATION AND THE DEFORMATION OF SOLIDS WITH A DISCRETE STRUCTURE

V. V. Bulatov

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Mechanical models and equations of state of solids with a discrete structure such as concretes, cement stones, rocks, etc., are discussed. Models are proposed describing the relaxation and deformation of the solids mentioned.

The mechanical models presented in the literature do not describe the behavior of solids with a discrete structure such as concretes, rocks, etc. ([1] and others). An analysis of the strain and creep curves of the materials mentioned indicates that a model should possess a number of the following properties.

1. An instantaneously applied load produces a corresponding deformation.

2. Under constant stress the strain increases with time, asymptotically approaching a limit which depends on the stress.

3. The limiting strain depends nonlinearly on the stress.

4. Up to a certain value of the stress (the elastic limit) the deformation of the body is elastic. Plastic-viscous strain begins beyond the elastic limit.

5. The increase in viscoplastic strain is accompanied by a simultaneous increase in elastic strain (Figs. 1 and 2).

Models have been developed which satisfy one or more of the conditions listed, but they do not satisfy all of them. Therefore, we propose a model of a body corresponding to the five requirements indicated. Its structural formula is

$$\Sigma = \mathrm{Pr} - \mathrm{K}. \tag{1}$$

Condition 1 is satisfied by the presence of an H element; the yield point and the appearance of plastic properties of the body are simulated by including a St.-V element (condition 4). Conditions 2, 3, and 5 are satisfied by including a K unit. Viscous properties are simulated by an N element and the asymptotic character of the ε vs t curves is achieved by a parallel content of H_2 and N elements. Condition 5 is satisfied by an appropriate disposition of H_1 and H_2 elements in the model proposed.

The equation of state of a Pr — K body is written in the form

$$\sigma(t) = E_1\varepsilon_1 + [E_2\varepsilon_2(t) + \eta\dot{\varepsilon}_2(t)\,\bar{\eta}(\sigma - \sigma_s)],$$
$$\sigma_s = E_1\varepsilon_T = A = \mathrm{const}, \tag{2}$$

$\bar{\eta}(\sigma - \sigma_s)$ is the Heaviside unit function,

$$\sigma < \sigma_s, \quad \sigma(t) = E_1\varepsilon_1(t),$$
$$\sigma \geqslant \sigma_s, \quad \sigma(t) = E_1\varepsilon_T + E_2\varepsilon_2(t) + \eta\dot{\varepsilon}_2(t) = A + E_2\varepsilon_2(t) + \eta\dot{\varepsilon}_2(t). \tag{3}$$

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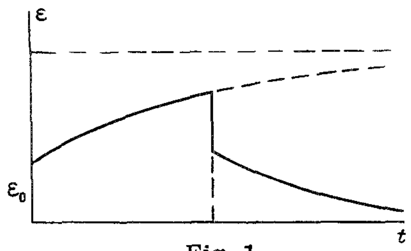

Fig. 1

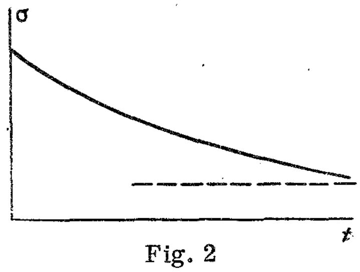

Fig. 2

Fig. 1. Time development of strain for a constant load which is later removed.

Fig. 2. Time variation of stress for constant strain.

For $\sigma(t) = \sigma_0 = \text{const}$ we have

$$\dot{\varepsilon}(t) + \frac{E_2}{\eta}\varepsilon(t) = \frac{\sigma_0 - A}{\eta} = B. \tag{4}$$

The solution of (4) for an initial strain $\varepsilon|_{t=0} = \varepsilon_0$ has the form

$$\varepsilon = \frac{B\eta}{E_2} + \left(\varepsilon_0 - \frac{B\eta}{E_2}\right)\exp\left(-\frac{E_2}{\eta}t\right), \tag{5}$$

i.e., for $\sigma = \text{const}$ the strain varies exponentially.

Thus, condition 2 is satisfied.

A fault of the proposed model is that it does not relax. A Pr — K medium best describes the creep process of rocks but is not suitable for the study of stress relaxation.

A medium which permits the study of relaxation and satisfies most of the requirements analyzed above is a P — Th body obeying a nonlinear relation between the limiting stress and strain [2]:

$$\varepsilon = D\sigma + F\sigma^2. \tag{6}$$

Then the equation of state in the one-dimensional case can be written in the form

$$\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \frac{1}{t_0}\left[\alpha\frac{\sigma}{E} + \beta\left(\frac{\sigma}{E}\right)^2 - \varepsilon\right], \tag{7}$$

$$\dot{\varepsilon} + \frac{1}{t_0}\varepsilon = \frac{1}{E}\dot{\sigma} + \frac{1}{t_0}\left[\alpha\frac{\sigma}{E} + \beta\left(\frac{\sigma}{E}\right)^2\right]. \tag{8}$$

Setting $\sigma = \text{const}$, we solve the homogeneous equation corresponding to (8).

Setting $\varepsilon|_{t=0} = \varepsilon_0$, we obtain

$$\varepsilon = C\exp\left(-\frac{t}{t_0}\right) = B + (\varepsilon_0 - B)\exp\left(-\frac{t}{t_0}\right) = \frac{1}{t_0}\left[\alpha\frac{\sigma}{E} + \right.$$
$$\left. + \beta\left(\frac{\sigma}{E}\right)^2\right] + \left\{\varepsilon_0 - \frac{1}{t_0}\left[\alpha\frac{\sigma}{E} + \beta\left(\frac{\sigma}{E}\right)^2\right]\right\}\exp\left(-\frac{t}{t_0}\right). \tag{9}$$

Condition (2) is satisfied.

For $\varepsilon = \text{const}$ we have

$$\frac{1}{E}\dot{\sigma} + \frac{\alpha}{t_0 E}\sigma + \frac{\beta}{t_0 E^2}\sigma^2 = \frac{\varepsilon}{t_0}. \tag{10}$$

Introducing the notation $\alpha/t_0 = b$, $\beta/t_0E = a$, and $\varepsilon E/t_0 = c$, we write

$$\dot{\sigma} + a\sigma^2 + b\sigma = c; \tag{11}$$

$$\dot{\sigma} + \left(\sqrt{a}\,\sigma + \sqrt{\frac{\alpha^2 E}{4t_0\beta}}\right)^2 = c + \frac{\alpha^2 E}{4t_0\beta}. \tag{12}$$

Setting $\alpha^2 E/4t_0\beta = d$ and $c + d = g$, we obtain

$$\dot{\sigma} + (\sqrt{a}\,\sigma + \sqrt{d})^2 = q. \tag{13}$$

Introducing the notation

$$\sqrt{a}\,\sigma+\sqrt{d}=z;\ \frac{dz}{dt}=\sqrt{a}\ \frac{d\sigma}{dt};\ \ \dot{\sigma}=\frac{\dot{z}}{\sqrt{a}} \tag{14}$$

and substituting into (13), we obtain

$$\dot{z}+\sqrt{a}\,z^2=\sqrt{a}\,q=k. \tag{15}$$

Equation (15) is a Riccati equation whose solution [3] for $\sqrt{a}k > 0$ has the form

$$z=\frac{z_0 a\sqrt{k}+k\,\mathrm{th}\,a\sqrt{k}\,t}{a\sqrt{k}+az_0\,\mathrm{th}\,a\sqrt{k}\,t}. \tag{16}$$

The curve passes through the point $(0,\ z_0)$; according to (14) we have

$$\sigma=\frac{z_0 a\sqrt{k}+k\,\mathrm{th}\,a\sqrt{k}\,t}{a\sqrt{ak}+az_0\,\mathrm{th}\,a\sqrt{k}\,t}-\sqrt{\frac{d}{a}}. \tag{17}$$

Here

$$z_0=\sqrt{a}\,\sigma_0+\sqrt{\frac{d}{a}}.$$

Equation (17) describes the relaxation of a solid.

The properties of solids depend to a considerable degree on such physical conditions as the difference between external and pore pressures and temperatures and the characteristics of liquid and gaseous media. It is a serious fault of the known equations of state that they do not contain these factors.

Further improvement of mechanical models of solids must obviously take account of the factors mentioned. The proposed models can be applied to construct generalized models of the stability and rupture of rocks and soils in the production of various mining engineering and construction operations.

NOTATION

Pr	is the Prandtl medium;
K	is the Kelvin — Voigt medium;
H	is the Hooke medium;
N	is the Newton medium;
St.-V	is the Saint-Venant medium;
σ	is the stress;
E_1	is the Young's modulus in the region of pure elastic deformations;
E_2	is the Young's modulus in the region of elasticoplastic deformations;
σ_S	is the elastic limit;
ε	is the strain;
ε_T	is the strain corresponding to σ_S;
η	is the coefficient of viscosity;
$\bar{\eta}$	is the Heaviside unit function;
t	is the time;
P — Th	is the Poynting — Thomson medium;
σ_0	is the stress at $t = 0$;

$E_1\varepsilon_T = A$; $(\sigma_0 - A)/\eta = B$;

D and F	are coefficients;
α and β	are dimensionless coefficients;

$t_0 = \eta/E$; $\alpha^2 E/4t_0\beta = d$; $c + d = g$; $\alpha/t_0 = b$; $\beta/t_0E = a$; $\varepsilon E/t_0 = c$; $\sqrt{a}q = k$.

LITERATURE CITED

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3. E. Kamke, Handbook of Ordinary Differential Equations [Russian translation], GIFML, Moscow (1971).